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Hydrodynamic conditions of transfer processes through a radial jet spreading over a flat surface

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Abstract—Fluid dynamics of a radially spreading liquid film originated by an ideal jet that falls onto a horizontal plate are studied approximately. Five regions of different hydrodynamic structures can be singled out here. The first one is that of the normal impingement of the jet against the plate, in which the flow essentially changes its direction. The second and the third regions correspond to laminar film flow before and after the emergence of the viscous boundary layer on the free surface of the film, respectively. The fourth region represents a zone in which a hydraulic jump takes place, where the film thickness drastically increases, and the fifth one is a region of calm gravitational spreading of the film up to the liquid running off the plate. Flow patterns within all the regions except that of hydraulic jump are considered on a basis of the Karman–Pohlhausen and Blasius methods and are conjugated in between. It is shown for the first time that the hydraulic jump on a sufficiently extended film owes its origin to the fact that the region with the viscous film flow induced by the initial jet momentum must come into contact with the region of the film which spreads under gravity. The results are obtained in a simple explicit form. They may lay a foundation for heat and mass transfer studies. A transfer problem is considered within the scope of the Karman–Pohlhausen method at an arbitrary Peclet number and asymptotically at high Peclet numbers with the help of the thin diffusional layer approximation.

1. INTRODUCTION

FILM FLOWS over solid surfaces, including those generated by impinging laminar and turbulent jets, are important for the problem of cooling hot bodies as well as for physico-chemical processing of metals and other solid materials. When dealing with such applications, one faces difficult situations of convective heat and mass transfer in a liquid film and needs a reliable basis for treating them to full advantage. This, in turn, requires a detailed knowledge about the velocity field within various film regions.

The theory of film flows of this type has long attracted significant attention. Its state has not changed considerably, however, for the last two or three decades and now it is substantially the same as presented in refs. [1–3]. Self-similar velocity fields specific to different flow regions have been found and, using the requirement of momentum conservation, an equation has been obtained which relates the film thickness after the hydraulic jump to that before it and also to physical and regime parameters. Notwithstanding this, the velocity fields are expressed in a form which is not tractable enough to be conveniently employed when studying heat or mass transfer problems. Besides, the very physical reason for the occurrence of the hydraulic jump remains obscure. This prevents the determination of other characteristics of the jump as well as of the conditions under which it actually makes its appearance.

The indicated features of the present-day theory hamper an effective study of manifold transfer processes to a considerable extent. It is evidenced by many attempts undertaken to this end (examples are to be found in refs. [4–6]). There are also some persistent discrepancies between theoretical predictions and experimental data, an example of which is supplied by the behaviour of the local film Nusselt number in the immediate vicinity of an impinging jet. In what follows, simple analytical expressions of the flow velocity in diverse parts of an axisymmetric film are obtained by using the approximate fluid dynamics methods. A similar approach has been previously applied in ref. [7] to film flow produced by an inclined plane laminar jet falling onto a horizontal plate. The hydraulic jump is proved to occur whenever the film spreads far enough. Its appearance happens to be a necessary condition for the transition from viscous thin film flow before the jump to the gravity-induced thick film flow regime in the region adjacent to the edge of the plate. Only laminar flow conditions are considered, but the main findings can be easily extended to flows generated by turbulent jets. The immediate effects of the specific features of film flow on the mass or heat transport intensity are demonstrated by means of both treating convective transfer at high Peclet numbers in the thin diffusional boundary layer approximation and using the approximate Karman–Pohlhausen method.

2. PHYSICAL MODEL AND BASIC EQUATIONS

Let us consider a radial film flow produced by a laminar vertical ideal liquid jet hitting a horizontal disk at its centre as sketched in Fig. 1. Within the

NOMENCLATURE

a	jet radius
c	concentration of admixture
D	diffusivity of admixture
F	function defined in equation (21)
Fr	Froude number
FR	quantity defined in equation (9)
g	gravity acceleration
H	dimensionless film thickness
h	dimensional film thickness
Nu	Nusselt number
Pr	Prandtl number
p	pressure
q	mass or heat flux density
R	disk radius
Re	Reynolds number
r	dimensional radial coordinate
Sc	Schmidt number
Sh	Sherwood number
T	temperature
t	dimensionless variable introduced in equation (22)
U	velocity at free surface
v	velocity
x	variable introduced in equation (30)
z	dimensional vertical coordinate.

Greek symbols

Δ	dimensionless thickness of boundary layer
δ	dimensional thickness of boundary layer
ζ	dimensionless vertical coordinate defined in equation (21)
η	dimensionless vertical coordinate
λ	heat conductivity
ν	kinematic viscosity of liquid
ξ	dimensionless radial coordinate
ρ	density
χ	thermal diffusivity
ψ	stream function.

Subscripts

0	initial values
1	point where hydrodynamic layer comes to free surface
T	point where thermal layer comes to free surface
r	radial component
z	vertical component
+	after hydraulic jump
−	before hydraulic jump
j	point of hydraulic jump
R	disk edge
w	wall.

Superscript

′	connected with the thermal boundary layer.

central part of the disk with a linear dimension of the order of initial jet radius a, there occurs an irrotational flow that transforms the jet into an axisymmetric divergent film spreading over the disk surface (region I). Through the action of viscous stresses, a boundary layer appears near the solid wall and continues to develop further until its external boundary reaches the free surface of the film (region II). After that, in region III, an entirely viscous flow develops throughout the whole film. The flow of such a type ceases before a sudden sharp increase in the film thickness,

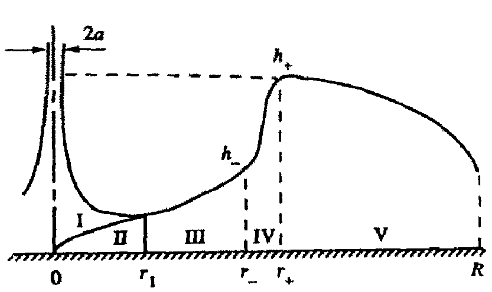

FIG. 1. A sketch of film flow (explanations are given in the text).

which is commonly referred to as a hydraulic jump zone. The width of this zone (region IV in Fig. 1) is rather small as compared with those of the other regions and may usually be ignored. Following the jump zone, there is another region V in which the film thickness gradually decreases until the liquid drains down the disk edge. Such a picture can be conceived as a result of the whole bulk of both pertinent experiments and theoretical concepts [1–3, 7].

Leaving the analysis of the flow inside region I, aside for the moment, we shall turn to regions II, III and V within which the film may be safely considered in the common thin fluid layer approximation. Boundary layer equations can be written as

$$v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}=\nu\frac{\partial^2 v_r}{\partial z^2}-\frac{1}{\rho}\frac{\partial p}{\partial r},$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}-g=0,\quad \frac{\partial(rv_r)}{\partial r}+\frac{\partial(rv_z)}{\partial z}=0, \qquad (1)$$

where ρ and ν are the density and kinematic viscosity of the liquid, g is the gravity acceleration. There are imposed common no-slip conditions at the disk surface ($z=0$), and a zero tangential stress condition either at the outer border $z=\delta(r)$ of the boundary layer or on the free film surface $z = h(r)$. The pressure must be assumed constant above the film, so that the expression $p = \text{const} + \rho g(h-z)$ follows from the second equation in set (1).

A self-similar solution of equations (1) was obtained in refs. [1, 2] neglecting the pressure gradient due to gravity. Velocity profiles were then expressed in terms of rather complicated functions of the dimensionless vertical coordinate $\eta = z/\delta(r)$ or $\eta = z/h(r)$. An alternative approach based on the use of Schwetz's approximate method was suggested in ref. [3]. It yields rather simple results for velocity fields in various film regions. Because of their complexity, the former profiles are inconvenient for studying transfer processes, whereas the latter are oversimplified and insufficiently accurate to be applied with confidence to those processes. For this reason here we prefer to use the well-known Karman–Pohlhausen polynomial method, according to which the radial longitudinal component of the liquid velocity is approximated as follows:

$$v_r(r,z) = \tfrac{3}{2}U(r)\left(\eta - \frac{\eta^3}{3}\right)$$

$$\eta = \frac{z}{h(r)} \quad \text{or} \quad \eta = \frac{z}{\delta(r)}. \tag{2}$$

Here $U(r)$ is either a varying velocity at the free surface or a constant velocity U_0 of the impinging jet and of the ideal film flow beyond the boundary layer. The profile in equations (2) satisfies identically the boundary conditions at $z = 0$ and $z = h(r)$ or $z = \delta(r)$.

Integrate the first equation in set (1) throughout either the boundary layer in region II or the whole viscous film in regions III and V with the help of the other equations in set (1) and (2). In the first case we get

$$\frac{39}{280}U_0^2\frac{1}{r}\frac{d}{dr}(r\delta) = \frac{3}{2}\frac{\nu U_0}{\delta} + g\delta\frac{dh}{dr} \tag{3}$$

and in the second case

$$\frac{272}{875}\frac{a^4U_0^2}{r}\frac{d}{dr}\left(\frac{1}{rh}\right) = -\frac{6}{5}\frac{\nu a^2 U_0}{rh^2} - gh\frac{dh}{dr}. \tag{4}$$

The condition of the total flow rate conservation leads to the following relationship between $\delta(r)$ and $h(r)$

$$h(r) = a^2/2r + 3\delta(r)/8, \tag{5}$$

which is relevant to the situation in region II. Within regions III and V we get from the same condition

$$U(r) = 4a^2U_0/5rh(r), \tag{6}$$

which was already used to formulate equation (4).

It is convenient to introduce dimensionless variables and Reynolds and Froude numbers

$$\{\xi, \Delta, H\} = \frac{Re^{1/3}}{a}\left\{\frac{r}{Re^{2/3}}, \delta, h\right\},$$

$$Re = \frac{U_0 a}{\nu}, \quad Fr = \frac{ga}{U_0^2}. \tag{7}$$

Then equation (3) yields a relation for determining $\Delta(\xi)$ in the form

$$\frac{39}{280}\frac{d}{d\xi}(\xi\Delta)^2 = 3\xi^2 + FR\left(\frac{3}{4}\xi^2\frac{d\Delta}{d\xi} - 1\right)\Delta^2 \tag{8}$$

and equation (4) gives

$$\frac{272}{875}\frac{d}{d\xi}(\xi H) = \frac{6}{5}\xi^2 + FR(\xi H)^3\frac{dH}{d\xi}$$

$$FR = \frac{Fr}{Re^{1/3}}. \tag{9}$$

Equations (8) and (9) form the main basis for subsequent calculations.

3. SOLUTION OF GOVERNING EQUATIONS

For the great majority of problems of practical interest, the last terms on the right-hand sides of equations (8) and (9) contain the small factor FR and may often be dropped out. In such a case, equations (5) and (8) result in

$$H(\xi) \approx \frac{1}{2\xi} + \frac{3}{8}\Delta(\xi)$$

$$\Delta(\xi) = \left(\frac{280}{39}\xi\right)^{1/2}, \quad \xi < \xi_1. \tag{10}$$

The boundary layer comes to the free surface at $\xi = \xi_1$, where ξ_1 must be evaluated from the equality $\Delta(\xi_1) = H(\xi_1)$. This, in view of equations (5) and (7), gives $H_1 = H(\xi_1) = 4/5\xi_1$ and, further, with account of equation (10),

$$\xi_1 = (78/875)^{1/3} \approx 0.447, \quad H_1 \approx 1.79. \tag{11}$$

Similarly, the solution of equation (9) without the term proportional to FR, which satisfies the condition $H(\xi_1) = H_1$ with $\xi > \xi_1$ gives

$$H(\xi) = \frac{175}{136}\xi^2 - \frac{C}{\xi}$$

$$C = \frac{175}{136}\xi_1^3 - H_1\xi_1 \approx -0.685. \tag{12}$$

The dimensionless thicknesses of both the boundary layer and the film are illustrated in Fig. 2. It can be concluded, first, that formulae (10) and (12) give a reasonably good approximation to exact solutions of equations (5), (8) and (9) when $\xi \leqslant 2$ and, second, that the use of approximate velocity profile (2) does not introduce a considerable error as against the self-similar solution of the hydrodynamic problem obtained in ref. [2]. The error can be proved not to exceed five percent within the whole indicated region.

When the last term on the right-hand side of equation (9) is taken into account, the derivative $dH/d\xi$ can be seen to tend to infinity as ξ approaches $\xi_\infty - 0$,

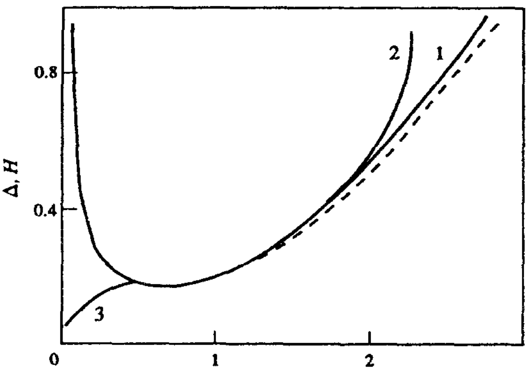

Fig. 2. Dimensionless film profiles: 1, formulae (10) and (12); 2, exact solution with account of gravity; 3, outer surface of the boundary layer; dashed line, result of ref. [2].

ξ_* being the only root of the equation $\xi_*^2 H^3(\xi_*) = (272/875)FR$. This means that a sharp increase in the film thickness is inevitable if the film flows far enough. Of course, solution (12) does not hold true and the original thin fluid layer equations (1) cease to be valid near the point $\xi = \xi_*$. Nevertheless, the above conclusion is indicative of the onset of a certain hydrodynamic crisis bound to occur at a sufficiently remote distance from the jet.

The crisis is evidently due to the development of the hydrostatic pressure at the solid plate along the spreading film of growing thickness. It is caused entirely by an increase in the weight of the film per unit area. The derivative $\partial v_r/\partial z$ near the plate turns to zero at $\xi = \xi_*$. This is similar to the condition which determines the detachment of a laminar boundary layer from an underlying solid surface. By analogy, it can be concluded that a reciprocal flow directed towards the jet should arise in the vicinity of the plate at this point. This flow must favour the formation of a vortex which ultimately results in carrying the liquid off the plate with an observable increase in the film thickness. Such an explanation is consistent with the hypothesis first formulated by Tani and Kurihara, according to which the hydraulic jump is an immediate consequence of an adverse pressure gradient due to gravity (see ref. [2]). Thus, however small the body gravity forces, they seem to be capable in the long run to significantly influence the film flow.

This inference is rather of a general character, since experiments witness the hydraulic jump to begin well before the point $\xi = \xi_*$ is reached, so that it is the region $\xi \ll \xi_*$ which is actually pertinent to real films. Then, as it can be deduced from Fig. 2, one is free to neglect an effect of gravity on film flow everywhere in front of the jump. If the jump originates at $\xi = \xi_-$, then the dimensionless film thickness $H_- = H(\xi_-)$ before the jump has to be approximately calculated from equation (12). The vortex jump region ends at $\xi = \xi_+$ where $H_+ = H(\xi_+) > H_-$. The quantities H_- and H_+ are still unknown.

Next to the jump, a viscous film flow is established again. The situation with $\xi > \xi_+$ is governed by equation (9), just like that ahead of the jump, that is, when $\xi < \xi_-$. However, the action of gravity cannot be now overlooked, since it is just that very reason which produces the flow. If the liquid smoothly flows down the edge of the disk, then the following requirement must be fulfilled:

$$\frac{dH}{d\xi} \to -\infty \quad \text{at} \quad \xi \to \xi_R = Re^{-1/3}\left(\frac{R}{a}\right), \tag{13}$$

R being the disk radius. It can be seen from equation (9) that this requirement is actually satisfied at

$$H_R = H(\xi_R) = \left(\frac{875}{272} FR\, \xi_R^2\right)^{1/3}$$

$$\approx 0.68\left(\frac{Re}{Fr}\right)^{1/3}\left(\frac{a}{R}\right)^{2/3}. \tag{14}$$

The dimensionless film thickness H is a monotonously decreasing function of ξ within the whole region considered. Numerical solutions of equation (9) at diverse values of ξ_R and FR are presented in Fig. 3. It is worth noting that similar solutions for a plane film are expressible in an analytical form (see ref. [7]).

Thus, the hydraulic jump region $\xi_- < \xi < \xi_+$ appears to be necessary to adjust the original film flow generated by an impinging jet to the regime of calm gravitational spreading of a comparatively thick liquid layer conditioned by the drain requirement, equation (13) or (14).

4. JUMP CONDITIONS

The length of the jump zone is commonly small as compared with those of the other flow regions, so that it is permissible to take approximately that $\xi_- \approx \xi_+$

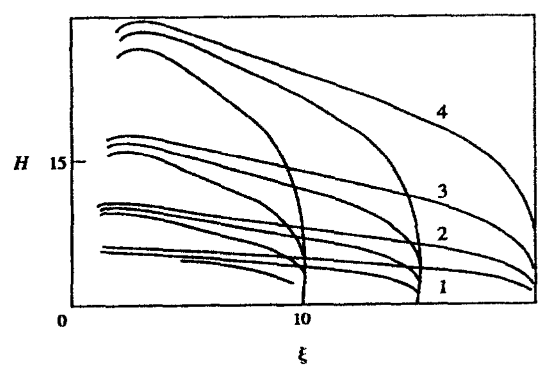

Fig. 3. Film profiles after hydraulic jump at different values of ξ_R and $\log(Fr/Re^{1/3}) = -2, -3, -4$ and -5 (curves 1–4, respectively).

$\approx \xi_j$ and to neglect the viscous friction inside the zone. The latter assumption allows for the momentum conservation equation to be used in a standard integral form [8]

$$\frac{1}{2}g(h_+^2 - h_-^2) = \int_0^{h_-} v_r^2\,dz - \int_0^{h_+} v_r^2\,dz, \qquad (15)$$

where h_- and h_+ refer to the film thickness before and immediately after the hydraulic jump, respectively. With the help of dimensionless quantities identified in equation (7), equation (15) yields

$$\frac{1}{2}FR(H_+^2 - H_-^2) = \frac{272}{875\xi_j^2}\left(\frac{1}{H_-} - \frac{1}{H_+}\right). \qquad (16)$$

This relation serves to find ξ_j, H_- and H_+ which is defined as $H(\xi_j)$ in conformity with equation (10) or (12) and with the solution of equation (9) in region V, respectively. This equation can be shown to have either a single physically suitable solution or none at all.

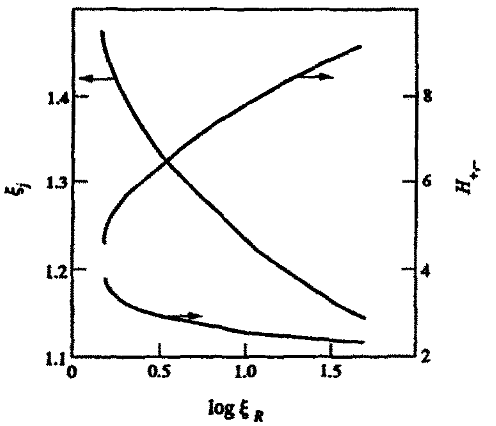

FIG. 5. Dimensionless jump coordinate and film thicknesses before and after the jump as functions of ξ_R at $Fr/Re^{1/3} = 0$; the curves terminate at the points at which solution of equation (16) ceases to exist.

Representative film profiles at $\xi_R = 4$ and different values of FR are plotted in Fig. 4. Dependencies of H_+ and H_-, as well as of ξ_j, on ξ_R are given at $FR = 0$ in Fig. 5. Finally, Fig. 6 illustrates the dependence of these dimensionless quantities on FR at $\xi_R = 4$.

Equation (16) has no solution when ξ_R either exceeds some maximal value, ξ_{max}, or is smaller than a certain minimal one, ξ_{min}, both of them depending on FR. That is why the curves of Fig. 5 terminate at some points corresponding to these extremal values. The case $\xi_R < \xi_{min}$ fits small disks, when the film flows down the disk without forming a hydraulic jump and the flow is of dynamic thin-film nature everywhere. To the contrary, when $\xi_R > \xi_{max}$ (large disks) the flow behaves according to the regime of calm gravitational spreading within the whole flow region, and the initial thickness of the film is higher, the larger the disk radius.

To make the properties of the hydraulic jump clearer, it seems plausible to consider, along with the curves in Figs. 4–6, a change which occurs in these properties with increase in the disk size under otherwise identical conditions. When the disk radius rises to a certain critical value, a weak hydraulic jump appears for the first time just nearby the disk edge. As the radius continues to grow, the jump gradually shifts towards the disk centre and its intensity enlarges progressively (which means that H_+ grows monotonously and simultaneously H_- monotonously decreases). At last, when another critical value of the disk radius is reached, the jump merges with the incident liquid jet and eventually vanishes. In the last case, the film profile is to be described by the curves of the type presented in Fig. 3.

From a physical point of view, such a behaviour of the hydraulic jump is caused by the growth of the overall hydraulic resistance to the film flow in the whole region up to the edge of the disk. A similar behaviour must surely be observed when the said growth results not from an increase in the disk size,

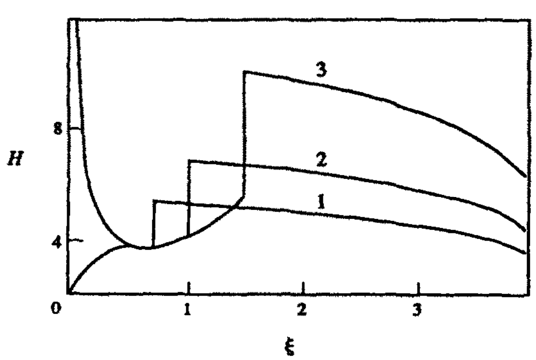

FIG. 4. Film profiles at $\xi_R = 4$ and $Fr/Re^{1/3} = 0.05$, 0.01 and 0.001 (curves 1–3, respectively).

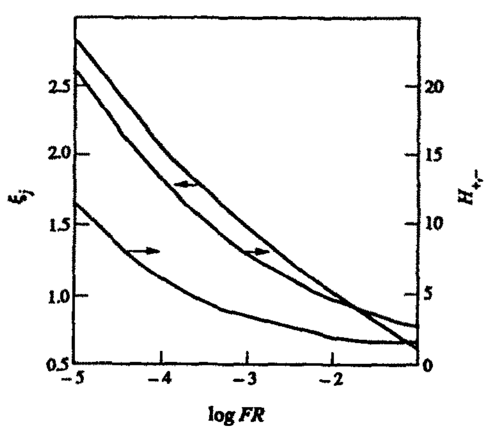

FIG. 6. The same quantities as in Fig. 5 as functions of $Fr/Re^{1/3}$ at $\xi_R = 4$.

but from some alternative reason. Thus, the hydraulic resistance of a disk of a given radius can be increased by means of using a sharp rim or putting a special concentric ring upon the disk edge, as it was the case in some experiments reported in ref. [3]. It follows from the above analysis that any of the above-indicated reasons must lead simultaneously to a displacement of the hydraulic jump closer to the jet, to a decrease in the thickness in the film that enters the jump region and to an increase of the thickness of the gravitationally expanding film that emerges from this region. All these expectations are completely confirmed by the experimental evidence of ref. [3].

Principally, the same behaviour of the hydraulic jump must be observed under unsteady conditions when a vessel is being filled up with liquid supplied by a jet which falls on the vessel bottom. We can easily convince ourselves of the correctness of this inference by placing a saucer under a water stream from a tap.

As regards the quantitative comparison of the developed theory with experiments, the situation is seriously hampered by the fact that the information, which is usually reported in the experimental works the present authors are aware of, is far from being sufficiently complete. Nevertheless, the results of such a correlation with the data of refs. [2, 3] are presented in Fig. 7, where the variable

$$w = \frac{1}{\pi^2} FR\xi_j H_j^2 + \frac{1}{2\pi^2 \xi_j H_j},$$

is employed which was previously suggested in ref. [2].

The agreement appears to be satisfactory. A slight discrepancy, which might be perceived while perusing Fig. 7, seems to be due, first, to the neglect of the finite dimension of the jump zone and, second, to the direct conjugation of the self-similar solutions relevant to different flow regions at common boundaries of these regions where the self-similarity should be lost.

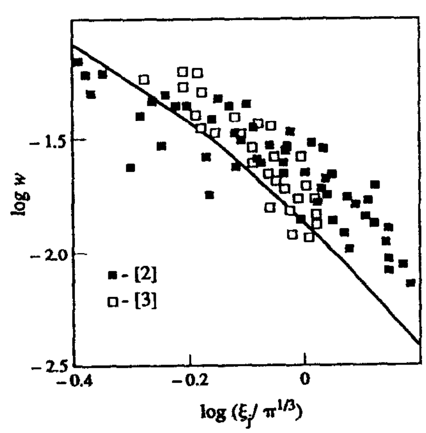

FIG. 7. Theoretical dependence of w on $\xi_j/\pi^{1/3}$ and experimental data of refs. [2, 3].

5. INITIAL REGION

We proceed further to an analysis of the jet deflection region (I in Fig. 1) in which the above thin-film theory does not hold. This region of radius $r_0 \sim a$ is of especial practical importance, since it provides for maximal heat or mass exchange between the plate and the radial liquid flow. To make a necessary correction, we suppose for simplicity that the film self-similar flow investigated above is established when $r > r_0$, whereas when $r < r_0$ we deal with an axisymmetric boundary flow. The latter flow can be described with the help of the well-known Blasius series discussed in detail in ref. [9]. By using a proper solution of the problem concerning a radial flow of ideal fluid generated by a vertical jet (see ref. [10]), we are able to write in the vicinity of the singular point of that flow

$$v_r = 0.44U_0 r/a, \quad v_z = -0.88U_0 z/a. \tag{17}$$

These formulae offer an opportunity to determine the first term of the Blasius series for the radial velocity component near the plate in the following form

$$v_r = 0.383Re^{1/2} U_0 rz/a^3. \tag{18}$$

This expression is assumed to be valid when $r < r_0$, where r_0 can be approximately evaluated by imposing the requirement that equation (18) should coincide with equation (2) at small z and precisely at $r = r_0$. Then

$$r_0 = (2.136)^{1/3} a \approx 1.288a. \tag{19}$$

Such an estimate is rather crude. However, it suffices to allow for an adequate description of heat or mass transfer efficiency near the critical point and, besides, may be easily improved by retaining subsequent terms of the Blasius series.

6. MASS TRANSFER AT HIGH SCHMIDT NUMBERS

To provide examples of application of the above findings to heat and mass transfer processes, we begin with studying a convective diffusion problem in the thin diffusional layer approximation. Suppose the film contains an admixture of original concentration c_0. The admixture is adsorbed at the plate so that its concentration at $z = 0$ equals zero. Then the standard formulation of the problem is [11]:

$$v_r \frac{\partial c}{\partial r} + v_z \frac{\partial c}{\partial z} = D \frac{\partial^2 c}{\partial z^2}, \quad c = 0, \quad z = 0;$$

$$c \to c_0, \quad z \to \infty. \tag{20}$$

with D being the diffusivity of the admixture.

The concentration differs from c_0 only within a very thin layer adjacent to the solid plate where the radial liquid velocity can approximately be presented as

$$v_r \approx u_0 F(\xi)\zeta, \quad \zeta = Re^{1/3}(z/a). \tag{21}$$

$$F(\xi) = \begin{cases} 0.383\, Re^{1/2}\xi, & 0 \leqslant \xi \leqslant \xi_0 \approx 1.288\, Re^{-1/3} \\ 0.560\xi^{-1/2}, & \xi_0 \leqslant \xi \leqslant \xi_1 \approx 0.447 \\ 1.2H^{-2}(\xi)\xi^{-1}, & \xi_1 \leqslant \xi \leqslant \xi_R \approx Re^{-1/3}(R/a). \end{cases}$$

Here the function $H(\xi)$ has a discontinuity at $\xi = \xi_j$ if ξ_j happens to be smaller than ξ_R (that is, if a hydraulic jump occurs).

Let us introduce the stream function ψ and a new dimensionless radial coordinate t by means of the relations

$$\psi = \frac{u_0 a^2}{2}\xi F(\xi)\zeta^2,$$

$$t = Da^2\, Re\sqrt{(2u_0)}\int_0^{\xi}\xi\sqrt{(\xi F(\xi))}\,d\xi. \quad (22)$$

Then by performing a standard calculation [11], we reduce problem (20) to that for a parabolic equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial \psi}\left(\sqrt{(\psi)}\frac{\partial c}{\partial \psi}\right),$$

the pertinent solution of which is to be written in the form

$$c = \frac{c_0}{1.17}\int_0^x \exp\left(-\tfrac{4}{9}x^3\right)dx, \quad x = \sqrt{(\psi)}t^{-1/3}. \quad (23)$$

The mass flux density is to be expressed as $q = D(\partial c/\partial z)$ at $z = 0$. This leads to a formula for the local Sherwood number

$$Sh = \frac{aq}{Dc_0} = 0.538(Sc\, Re)^{1/3} \times \left[\int_0^{\xi}\xi\sqrt{(\xi F(\xi))}\,d\xi\right]^{-1/3}\sqrt{(\xi F(\xi))} \quad (24)$$

$$Sc = \nu/D \gg 1, \quad \xi < \xi_j,$$

with the Reynolds number being identified in equation (7). This formula holds up to the hydraulic jump at $\xi = \xi_j$.

An expression for Sh after the jump can be derived in the same manner as before if one assumes that complete mixing is accomplished within the jump region. Then, a new diffusional boundary layer begins to develop when $\xi > \xi_j$ and, consequently,

$$Sh = 0.538\frac{c_0'}{c_0}(Sc\, Re)^{1/3} \times \left[\int_{\xi_j}^{\xi}\xi\sqrt{(\xi F(\xi))}\,d\xi\right]^{-1/3}\sqrt{(\xi F(\xi))}, \quad \xi > \xi_j, \quad (25)$$

where c_0' is the mean admixture concentration in the film after the jump which is fully determined by the mass balance condition with account of the admixture adsorption in the region before the jump.

In the vicinity of the critical point, one gets from equations (21) and (24)

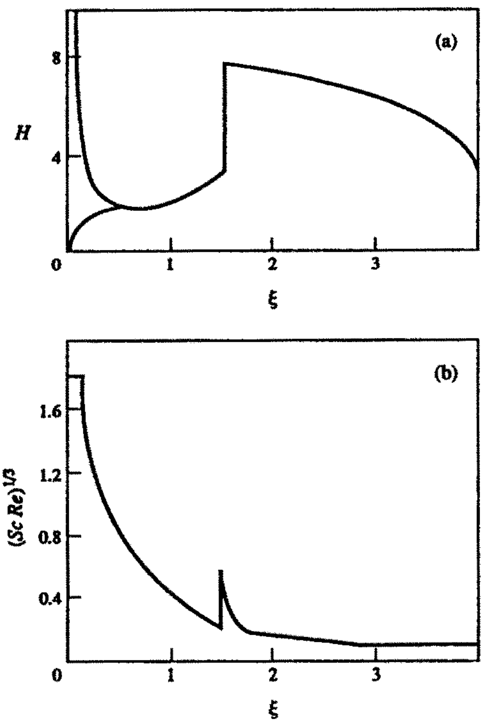

FIG. 8. Dimensionless film thickness (a) and local Sherwood number in the thin diffusional layer approximation (b) at $\xi_R = 4$, $Re = 1000$, $Fr/Re^{1/3} = 0.001$.

$$Sh = 0.538(Sc\, Re)^{1/3}(1.149\sqrt{(Re)})^{1/3} = 0.564 Sc^{1/3} Re^{1/2}. \quad (26)$$

Formulae (24)–(26) are illustrated in Fig. 8 for a particular case of film flow at $c_0' \approx c_0$. It can be seen that the mixing inside the jump region and the resulting renewal of the diffusional layer give rise to the maximum Sherwood number at $\xi = \xi_j$, after which Sh again begins to decrease with ξ. Formula (26) presents the upper limit of the local Sherwood number which is attained nearby the falling jet. The constancy of Sh at small ξ is stipulated by the fact that the liquid velocity at the outer edge of the hydrodynamic boundary layer is by no means constant within the initial region, but gradually increases from zero to u_0 and may be approximated at small r/a with the help of equation (17). This is quite compatible with numerous observations (see, for example, ref. [6]). If the special structure of the initial region were not taken into account, the local Sherwood number would indefinitely grow when $r \to 0$, as it is shown by the dashed line in Fig. 8.

7. HEAT TRANSFER AT AN ARBITRARY PRANDTL NUMBER

The above calculation scheme is quite suitable for studying mass transfer processes when the Schmidt number exceeds unity to a considerable extent. However, it becomes inapplicable to heat transfer problems when the Prandtl number is either larger or smaller than unity. An approximate model can be brought into action by using a method similar to that employed for the thin liquid layer flow.

For the sake of definiteness, we consider either cooling or heating of a solid plate maintained at a constant temperature T_w by a liquid film of an original temperature T_0. It is convenient to write the relevant convective heat conductivity problem in the form which is somewhat alternative to that in equation (20)

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r T)+\frac{\partial(v_z T)}{\partial z}=\chi\frac{\partial^2 T}{\partial z^2},$$

$$T=T_w, z=0; \quad T=T_0, z=\delta'(r), \tag{27}$$

where χ is the thermal diffusivity and $\delta'(r)$ is the thickness of the thermal boundary layer inside which the temperature differs from T_0. This quantity is unknown so far. Without loss of generality, T_0 may be taken equal to zero.

In compliance with the general idea of the Karman–Pohlhausen method, we assume

$$T=T_w\left[1-\frac{3}{2}\left(\eta'-\frac{\eta'^3}{3}\right)\right], \quad \eta'=\frac{z}{\delta'(r)}, \tag{28}$$

which in essence is similar to the approximation of the velocity profile in equation (2).

An equation for $\delta'(r)$ is to be obtained by means of integrating equation (27) over the layer thickness with account of equations (28) and (2) and of the fact that $v_r = u_0$ when $\delta(r) < z < h(r)$. Two cases should be distinguished when $\delta' < \delta$ $(Pr < 1)$ or $\delta' > \delta$ $(Pr > 1)$. In the first case we arrive at an equation

$$\frac{1}{\xi}\frac{d}{d\xi}\left\{\xi\Delta'\left[1-\frac{\Delta}{\Delta'}+\frac{2}{5}\left(\frac{\Delta}{\Delta'}\right)^2-\frac{1}{35}\left(\frac{\Delta}{\Delta'}\right)^4\right]\right\}=\frac{4}{\Delta' Pr}, \tag{29}$$

$$\Delta'=(\delta'/a)\,Re^{1/3}, \quad Pr=\nu/\chi\leqslant 1.$$

It is already known that $\Delta = C\sqrt{(\xi)}$ and $C=(280/39)^{1/2}$. Let Δ' be equal to $C'\sqrt{(\xi)}$, then we get from equations (29) an algebraic equation for the ratio $x = C'/C$

$$(1-x+\tfrac{2}{5}x^2-\tfrac{1}{35}x^4)=\frac{13}{35}\frac{x^2}{Pr}, \quad Pr\leqslant 1 \tag{30}$$

which has the root $x = 1$ precisely at $Pr = 1$.

It is easy to see that $x \geqslant 1$, and the outer boundary of the thermal layer comes to the free surface of the film when $\xi_T \leqslant \xi_1$ (that is, before this happens with the hydrodynamic boundary layer). By taking into account equation (5), we get an equation to determine ξ_T

$$\xi_T=\left(\frac{39}{280}\right)^{1/3}\left(\frac{4}{8/x-3}\right)^{2/3}, \tag{31}$$

where x stands for the relevant root of equation (30).

When $\delta' \geqslant \delta$ $(Pr \geqslant 1)$, instead of equation (30) we get

$$x^2\left(x-\frac{x^3}{14}\right)=\frac{13}{14\,Pr}, \quad Pr\geqslant 1, \tag{32}$$

which gives $x = 1$ at $Pr = 1$. In this case, the hydrodynamic layer comes to the free surface before the thermal layer and $x \leqslant 1$. The value of x determines $\Delta'_1 = \Delta'(\xi_1) = xH_1 \approx 1.79x$ (see equation (11)) at $\xi = \xi_1$.

In the region of viscous flow, we are able to derive, in quite the same manner as before, the following equation and initial condition:

$$\frac{1}{\xi}\frac{d}{d\xi}\left\{\xi\Delta'\left[\frac{\Delta'}{H}-\frac{1}{14}\left(\frac{\Delta'}{H}\right)^3\right]\right\}=\frac{12.5}{Pr}\frac{H}{\Delta'}\xi, \tag{33}$$

$$\xi>\xi_1: \quad \Delta'(\xi_1)=1.79x(Pr), \quad Pr>1.$$

Here, by introducing a new variable $X = \Delta'/H$, we obtain an equation for X with $X(\xi_1) = x$ which can be integrated numerically. The dimensionless coordinate ξ_T of the point at which the thickness of the thermal boundary layer becomes equal to the film thickness has then to be found from the equality $X(\xi_T) = 1$. The dependence of ξ_T on Pr is shown in Fig. 9. This quantity goes to infinity as Pr tends to 5. If $Pr > 5$, the outer boundary of the thermal layer never comes to the free surface, since the thickness of the film in the region preceding the hydraulic jump grows faster than that of the thermal layer.

The heat flux to the solid plate and the corresponding local Nusselt number are as follows:

$$q=-\lambda\frac{\partial T}{\partial z}\bigg|_{z=0}=\frac{3}{2}\frac{\lambda T_w}{a}\frac{Re^{1/3}}{\Delta'},$$

$$Nu=\frac{qa}{\lambda T_w}=\frac{3}{2}\frac{Re^{1/3}}{\Delta'}. \tag{34}$$

The quantity $Nu/(Re\,Pr)^{1/3}$ as a function of ξ in the region before the jump at different Pr's is illustrated in

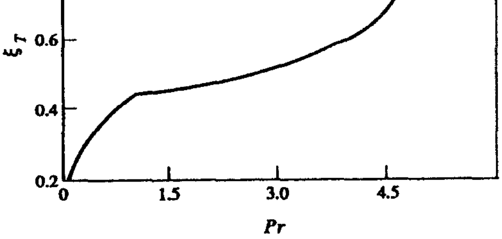

FIG. 9. Dimensionless coordinate at which the thermal boundary layer comes to the film free surface as a function of Prandtl number.

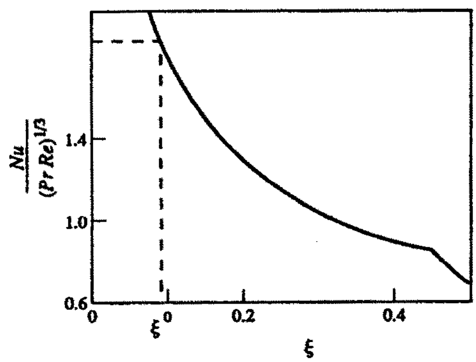

FIG. 10. Local Nusselt number before the jump at $Pr = 3$; dashed line gives a value pertinent to the initial flow region as explained in the text.

Fig. 10. This quantity is restricted from above by the value $0.564\ Re^{1/6}$ which gives $Nu/(Re\ Pr)^{1/3}$ within the initial flow region (see equation (26)). It means that a real dependence of $Nu/(Re\ Pr)^{1/3}$ on ξ at a given Pr has to be approximately composed of a segment of that line parallel to the ξ-axis when $\xi < \xi^0$ (Re, Pr) and a part of the corresponding curve of Fig. 10 when $\xi > \xi^0$, ξ^0 being the coordinate of intersection of these curves, as is conventionally shown by the dashed line.

8. CONCLUDING REMARKS

The main issue of this paper consists in that it provides a convenient tool to investigate manifold transport processes of interest in thin liquid films spreading over horizontal solid plates. The derived representations of the velocity profile within diverse regions of a film happen to be simple enough to ensure their immediate use while studying convective transfer problems. This offers an opportunity to get tractable conclusions concerning the relevant distribution of the local Nusselt or Sherwood number over the plate without recourse to elaborated numerical methods, as it is conclusively proved by the simple examples considered above.

The same approach can readily be generalized to give a sufficiently simple analytical description of film flow of another origin. For instance, it is not difficult to obtain results pertaining to a film generated by an inclined plane or axisymmetric jet falling onto an inclined surface when gravity contributes to either acceleration or deceleration of the flow. A film originated by a plane inclined jet on a horizontal plate has been treated in ref. [7]. It is also easy to see that the method is well applicable to films impinging upon a moving surface, which are of great practical significance when cooling rolled metals and in some other engineering designs. Moreover, turbulent jets and films could be treated in much the same way if one cares to incorporate into the analysis an empirical dependence of the effective viscosity due to turbulence, as it was earlier done in refs. [1–3].

We conclude with a brief indication concerning the specific features of the film cooling of a plate which is overheated above the boiling temperature of a coolant. Outside the region, in which the coolant first hits the plate and a vapour sublayer may occur, one might expect the intensive evaporation from the film free surface to cause a substantial loss of liquid and so to affect the film hydrodynamics. In general, such an expectation is correct when $Pr < 5$. The smaller the Prandtl number, the earlier the boiling temperature establishes itself at the free surface and the greater is an influence of evaporation. That influence is capable, in particular, to prevent the formation of a hydraulic jump. However, it is not so for liquids with $Pr > 5$ when the thermal boundary layer has no time to reach the free surface and the temperature at the latter remains equal to an original temperature of the liquid coolant. Then the evaporation does not generally play any considerable role and film properties are practically the same as if there were no heat transfer process at all. The significance of the last notion is evident because the Prandtl number of the most common coolant—water—is certainly larger than the indicated critical value of five.

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